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Flocks and Formations

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Given a large number (the "flock") of moving physical objects, we investigate physically reasonable mechanisms of influencing their orbits in such a way that they move along a prescribed course and in a prescribed and fixed configuration (or "in formation"). Each agent is programmed to see the position and velocity of a certain number of others. This flow of information from one agent to another defines a fixed directed (loopless) graph in which the agents are represented by the vertices. This graph is called the communication graph. To be able to fly in formation, an agent tries to match the mean position and velocity of his neighbors (his direct antecedents on the communication graph) to his own. This operation defines a (directed) Laplacian on the communication graph. A linear feedback is used to ensure stability of the coherent flight patterns. We analyze in detail how the connectedness of the communication graph affects the coherence of the stable flight patterns and give a characterization of these stable flight patterns. We do the same if in addition the flight of the flock is guided by one or more leaders. Finally we use this theory to develop some applications. Examples of these are: flight guided by external controls, flocks of flocks, and some results about flocks whose formation is always oriented along the line of flight (such as geese).

KEY WORDS: Communication graphs; directed graph laplacians; stability of formations; linear feedback and control; systems of ordinary differential equations.

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1. INTRODUCTION

In observing large schools of fish or flocks of birds, what strikes us most is their capability of maneuvering while maintaining the pattern formed by their bodily positions (up to translation and/or rotation). If the flock is large enough, it dawns on us that the animals cannot possibly keep track of the positions and velocities of all the others; on contrary, they must execute a local algorithm that enables them to stay "in formation" while the flock as a whole executes its motion or maneuver. In fact, biologically based models studied in ref. 20 indicate that animals do not keep track of more than something on the order of ten others. This algorithm must be such that the configuration is stable against perturbation (such as sudden wind, etcetera).

While still little appears to be known about the algorithm that animals actually employ to fly (or run, or swim) in stable configurations (but see refs. 16, 20), neighbor-based algorithms are now employed to control the movement of groups of artificial agents (robots, drones, see refs. 10, 13, 15, 23, 25). In order to see how the need for such an algorithm may arise, we mention the following taken from the literature (see ref. 7). Suppose that many autonomous (not steered by an external source) vehicles find themselves in an enclosed space with only one relatively narrow exit. The object is to get all the vehicles to leave the space through the exit as quickly as possible, with a minimum of computation on the part of the vehicles, and without collisions. One way to achieve this is to have the vehicles move a circular formation first, then have the vehicles exit the through the opening one after the other. We end Section 8 with a simulation of a movement reminiscent of this problem.

The part of the problem we address is how can one achieve that autonomous agents move towards a formation and, once formation is attained, how can we ensure it is stable against perturbations. There are various approaches to achieve this. One of the first ones led to "boids".[24] In many of these applications one is not necessarily preoccupied with actual physical objects, but rather a numerical simulation. In ref. 26, previous approaches were greatly simplified, but again the equations are not plausible for physical agents (they are more interested in phase transitions). Fax and Murray[4,5] were among the first to write down equations that were meant to govern actual physical entities. Our exposition follows along these lines.

An area of research which is closely related to the theme of this paper is that of consensus seeking autonomous agents. In this case agents achieve consensus if their associated variables converge to a common value. In refs. 17, 18, 21, 23, the convergence to consensus is proved under a variety of circumstances. However, in our case the agents, being physical objects, do not want to coalesce to the same position but to a position a fixed distance away from its neighbors, and they must satisfy Newton's Law.

The algorithm we investigate (following refs. 11, 12, 3) in this paper has the following ingredients. The individual agents are identical (all do the same computation). Each individual agent complies with Newton's Law in that it changes position and velocity by applying a force or thrust, which causes it to accelerate in the desired direction. Furthermore, in order to compute the thrust each agent may use as its input only the following data: its own position and velocity relative to an absolute coordinate system and those of a small (and fixed) collection of neighbors. It will turn out that in fact only the position and velocity of the others *relative* to it will be used except when we wish the flock to accelerate (see the equations at the end of Section 2). In addition the computation requires the desired position of each agent in the formation. (Again, the aforementioned equations show that we only require the relative positions.) Finally, the algorithm that computes the thrust is only to use affine combinations of the above quantities, and must, at least for a large collection of initial conditions, cause the whole system to converge to an in formation orbit, that is: an orbit of the flock in which all the relative positions have the desired value.

In the next section we will consider such a model. We will address two main issues. The first is to decide exactly what kind of coherent motions the flight of the flock converges to if we choose an appropriate linear feedback. It turns out that this depends on the topology of the graph whose vertices represent the agents and whose directed edges (i, j) represent the relation: agent i sends information to agent j in the flock. Thus in Section 3 we discuss some notions of graph theory. The first main result, (Theorem 4.4), which specifies the set of asymptotic orbits of the flock, is a generalization of results obtained by refs. 5 and 12.

The second main issue is essentially new and addresses the question how to arrange the algorithm that computes the force or thrust applied by each agent in such a way that the "center of mass" of the flock follows a specified orbit *and* the configuration is stable against perturbations throughout the orbit (see Theorem 5.2). In ref. 12, this issue was addressed only in the case linear acceleration is desired.

In the remaining sections we simplify the stability calculation for hierarchies of graphs, extending earlier results of ref. 27 (see Section 7), and modify the notion of in formation to a biologically more reasonable one that involves orientable configurations (Section 8). The equation that governs the evolution of this system is nonlinear. The last section summarizes the main lines of thought in this paper, now with the benefit of being able to use the language developed in the course of this work.

This exposition depends on a theorem stated here (as Theorem 3.6) but proved in a companion paper[3] since its proof requires quite a few lemmas unrelated to the main argument here. With this exception, the theory developed here is self-contained.

2. THE LINEAR MODEL

The individual agents are assumed to move in $\mathbb{R}^d$, so that the dynamical coordinates (position and velocity) of each agent constitute a point in the tangent bundle $T\mathbb{R}^d \cong \mathbb{R}^{2d}$. In physical applications (such as buffalo or drones) d will usually be 2 or 3. The agents are numbered, say, from 1 through N. It is convenient to think of the coordinates of the entire flock as the graph of a function

$$C : \{1, \dots, N\} \to \mathbb{R}^{2d}$$

in the product $X \equiv \{1, \dots, N\} \times \mathbb{R}^{2d}$. Indeed we will end up writing a general position in X as a (tensorial) sum of the positions of each agent (see Equations 2.1 and 2.2).

We now choose coordinates in X. A point in $z \in X$ is specified by a column vector of $2dN$ real numbers as follows. The position and velocity of agent j are given by the entries numbered $2d(j-1)+1$ through $2dj$. The first of these real numbers specifies the first of its position coordinates (the x-coordinate) and the second specifies the first of the velocity coordinates. The third entry is the second position coordinate, and so on alternatingly.

It turns out that the Kronecker product ($\otimes$) provides us with the adequate description of the space X. Using this product, we can write an arbitrary point $z \in X$ as

$$z = \sum_{i=1}^{N} e_i \otimes \rho_i, \tag{2.1}$$

where e_i is an N-vector with a single 1 in the k-th entry as its only nonzero entry and ρ_i is the position-velocity vector of the i-th agent. Alternatively, if we wish to specify the position x_i and velocity v_i of the individual agent separately, we can write

$$z = \sum_{i=1}^{N} e_i \otimes \left(x_i \begin{pmatrix} 1 \\ 0 \end{pmatrix} + v_i \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right). \tag{2.2}$$

Each agent is assigned a desired position $h_i \in \mathbb{R}^d$ in the flock. (We can also assign a desired velocity, but the only choice for a stabilizable in formation state turns out to be a velocity that is the same for all agents, see Section 5.) We thus obtain a vector h as follows:

$$h = \sum_{i=1}^{N} e_i \otimes h_i \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

This vector is the desired configuration vector.

Definition 2.1. The orbit $\phi : \mathbb{R} \to X$ of the flock is said to be *in formation* if it is given by

$$\phi(t) = h + \sum_{i=1}^{N} e_i \otimes \alpha = h + \mathbf{1}_N \otimes \alpha,$$

for some function $\alpha : \mathbb{R} \to \mathbb{R}^{2d}$ of the form

$$\alpha = q \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + p \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad p = \frac{dq}{dt},$$

where $\mathbf{1}$ denotes the N-dimensional vector of all ones.

Each agent is allowed to see only some of its colleagues (its neighborhood) and applies the same linear feedback as the others. This neighborhood does not vary. The agent continually averages its neighbors' positions and velocities, then subtracts that from its own. It then compares this with the *desired* outcome of that calculation (namely, the same operation performed on the h_k). Thus if agent k has agents i and j as its neighbors, it calculates the d-dimensional vector $(x_k - h_k) - 1/2(x_i - h_i) - 1/2(x_j - h_j)$ and another one, $v_k - 1/2v_i - 1/2v_j$. The operation that does this for each agent will be called the (directed) Laplacian. The agent is then allowed to compute linear combinations of the components of this $2d$-dimensional vector and use these to determine its thrust. Finally, we allow the agent to correct its orbit using linear combinations of its own position-velocity vector.

As an example, consider a flock in $\mathbb{R}^3$ so that agent i has position (x_i, y_i, z_i), velocity (u_i, v_i, w_i), and desired position $(h_{x,i}, h_{y,i}, h_{z,i})$.

Assume that agent 1 sees only agents 2 and 3. The equations of motion for agent 1 become (with some further assumptions, see below).

$$
\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{u}_1 &= au_1 + f\left((x_1 - h_{x,1}) - \frac{1}{2}(x_2 - h_{x,2} + x_3 - h_{x,3})\right) + g\left(u_1 - \frac{1}{2}(u_2 + u_3)\right) \\
\dot{y}_1 &= v_1 \\
\dot{v}_1 &= av_1 + f\left((y_1 - h_{y,1}) - \frac{1}{2}(y_2 - h_{y,2} + y_3 - h_{y,3})\right) + g\left(v_1 - \frac{1}{2}(v_2 + v_3)\right) \\
\dot{z}_1 &= w_1 \\
\dot{w}_1 &= aw_1 + f\left((z_1 - h_{z,1}) - \frac{1}{2}(z_2 - h_{z,2} + z_3 - h_{z,3})\right) + g\left(w_1 - \frac{1}{2}(w_2 + w_3)\right)
\end{aligned}
$$

These equations are invariant under Galilean transformations only if $a=0$. The parameter a and other similar parameters in this model (the matrix A_4 in the remark after Theorem 4.2) are parameters that can used by a central authority to 'steer' the orbit of the flock as a whole. A few examples of this will be discussed in Section 5.

In a more compact notation using the Kronecker product for matrices as well, we can write the above equations of motions for the whole system very succinctly in the following way.

Definition 2.2. With the above conventions, the equations of motion of the flock are:

$$
\dot{z} = I_N \otimes Az + L_N \otimes K(z-h). \tag{2.3}
$$

Here I_N is the $N \times N$ identity matrix and L_N is the $N \times N$ (directed) Laplacian matrix (see Section 3). The matrices A and K are assumed to be constant. The matrix K may be chosen to stabilize in formation orbits (see below).

The matrix K is essentially a linear filter. To obtain the equations of the above example (these are the equations studied by ref. 12), we chose A and K to have a particularly simple form, namely:

$$
A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & a \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ f & g & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f & g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & f & g \end{pmatrix}.
$$

The origin of the matrix K in equation 2.3 is in a control theoretic formulation of this problem, of which an extensive discussion can be found in ref. 5. The thought underlying the last term of equation 2.3 is that each agent performs two computations. Agent k filters its data by applying the $2d \times 2d$ matrix K to its own $2d$ components of the vector $z - h$ (the vector $z_k - h_k$), *and* it applies the k-th row of the Laplacian L_N to the flock (this computation involves only data from the agents it sees directly). That said, we should remark that the order in which these operations are executed is irrelevant, since:

$$(L_N \otimes I_{2d})(I_N \otimes K) = (I_N \otimes K)(L_N \otimes I_{2d}).$$

Thus we may think of the agents as computing the Laplacian before filtering their output. Since the output of the computation gives the acceleration, we can think of K as a feedback. This is also the approach taken by ref. 5.

In Section 8, a model will discussed in which each h_i is rotated by the angle in which agent i is flying. This has the advantage that the formation as a whole can orient itself. However, the resulting equation is nonlinear.

As a final remark in this section, we should point out that the main conclusions are valid if we replace the above Laplacian by a weighted version of it: the diagonal entries equal 1, all other entries are non-positive, and row-sums are zero. We do not pursue this here in the interest of readability (but see ref. 3).

3. THE LAPLACIAN IN GRAPH THEORY

Throughout this paper we focus on the problem of a fixed communication graph. While in the biological setting the 'neighbors' change, there is a substantial analysis to be done in this case as the next few sections demonstrate. Moreover, in the context of controlling robots, unmanned vehicles, or satellites, a fixed communication is quite natural and suitable for designing coordination control algorithms.

In this section we give the definitions of some graph theoretic notions relevant to the development of the theory of the model discussed in the Section 2. We mention a characterization of the (right) eigenspace associated with the eigenvalue zero for a Laplacian on an arbitrary graph (Theorem 3.6). At the end of the section we also discuss an example that we will use throughout the text.

As was discussed in the previous sections, each agent i receives information from a fixed collection of agents distinct from it. This collection is denoted by $\mathcal{N}(i)$, the neighborhood of i.

Definition 3.1. An agent at v is called a *leader* if $\mathcal{N}(v) = \emptyset$.

Definition 3.2. The *communication graph* G is the (directed and loopless) graph whose i-th vertex represents agent i and that has a directed edge from j to i if and only if $j \in \mathcal{N}(i)$.

Recall that a graph is called loopless if no edge connects a vertex to itself. The in-degree matrix D and the adjacency matrix Q are given as follows. The matrix D is diagonal and its *only* non-zero entries are given by

$$d_{ii} = \mathrm{card}(\mathcal{N}(i)) \quad \text{if} \quad \mathcal{N}(i) \neq \emptyset.$$

Denote by D^+ the diagonal matrix whose *only* non-zero entries are given by

$$d_{ii} = 1/\mathrm{card}(\mathcal{N}(i)) \quad \text{if} \quad \mathcal{N}(i) \neq \emptyset.$$

(This is sometimes called the pseudo-inverse of D.) The adjacency matrix Q is the matrix whose *only* non-zero elements are given by

$$q_{ij} = 1 \quad \text{if} \quad j \in \mathcal{N}(i).$$

Definition 3.3. The *directed Laplacian* associated with the graph G is given by

$$L \equiv D^+(D - Q).$$

A few remarks are in order. With this definition the Laplacian on a regular 2-dimensional lattice is a *negative* multiple of the standard discretization of the Laplacian (which would correspond to $Q - D$). (There are other conventions possible, the choice essentially dictated by convenience. For instance: in ref. 12, we chose to follow a different convention, namely: $L = D - Q$.)

For future reference we observe at this point that if agent i is a leader, then the i-th row of the Laplacian consists of zeroes.

We are interested in the spectrum of this Laplacian. Many things are known about the spectrum of the undirected Laplacian (see ref. 7 for an in-depth treatment, and ref. 3 for some facts relevant to the current situation). The most familiar statement of these is undoubtedly that if G is undirected and connected the spectrum L is real and contained in $[0, 2]$ and moreover the eigenvalue zero has algebraic and geometric multiplicity

1. Its associated (right) eigenvector is (a multiple of) $\mathbf{1}$, the vector whose entries are equal to 1.

In ref. 12, we proved an extension of this, namely if G is a loopless directed graph, then zero is an eigenvalue with algebraic and geometric multiplicity 1 if and only if G has a rooted, directed spanning tree. The latter means that G has a vertex v such that there is a directed path from v to every other vertex in G. In Theorem 3.6 below, we will give the multiplicity of the zero eigenvalue and characterize its associated eigenvectors for arbitrary directed graphs. The proof of this result will be published in a forthcoming paper.[3] First we need some notation.

Given any vertex v in G, the reachable set $\mathcal{R}(v)$ is the union of v and the collection of vertices j so that there is a directed path from v to j.

Definition 3.4. A *reach* R is a maximal reachable set in G, that is: a collection R is a reach if it equals $\mathcal{R}(v)$ for some $v \in G$ and if there is no $\ell \in G$ so that $\mathcal{R}(v) \subset \mathcal{R}(\ell)$ (properly).

Notice that whenever a v is a leader, $\mathcal{R}(v)$ must be a reach. Further, a strongly connected graph has exactly one reach. However, a weakly connected graph may have more than one reach and if that is the case the reaches cannot all be mutually disjoint. It is not hard to see that one can always write a finite graph G as a finite union of reaches $G = \cup_{i=1}^{k} R_i$, and that for a given graph G the collection $\{R_1, \ldots, R_k\}$ of reaches is unique (up to permutation).

Definition 3.5. Let $\{R_1, \ldots, R_k\}$ denote the reaches of G. The *common part* C_i of the i-th reach R_i is defined as

$$C_i \equiv R_i \cap \{\cup_{j \neq i} R_j\}.$$

The complement of the common part C_i in the reach R_i will be referred to as the *exclusive part* H_i.

Note that for any given reach R_i, its common part C_i may be empty, but the exclusive part H_i is not.

As an example we give the adjacency matrix Q and the associated (directed) Laplacian of a connected graph with 5 vertices whose reaches are given by $R_1 = \{1, 2, 3, 4\}$ and $R_2 = \{3, 4, 5\}$ with exclusive parts $H_1 = \{1, 2\}$ and $H_2 = \{5\}$.

$$Q = \begin{pmatrix} 0&0&0&0&0 \\ 1&0&0&0&0 \\ 1&0&0&1&0 \\ 0&0&1&0&1 \\ 0&0&0&0&0 \end{pmatrix} \text{ and } L = D^{+}(D - Q) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1/2 & 0 & 1 & -1/2 & 0 \\ 0 & 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The Laplacian acts from the left on vectors. We will consider these vectors as functions *from* the vertices of the graph G *to* $\mathbb{C}$.

Theorem 3.6. Let L be the (directed) Laplacian matrix associated with a loopless graph G with N vertices and exactly k reaches. Then the eigenvalues of L are contained in the closed unit disk in the complex plane centered on 1. The algebraic and geometric multiplicity of the eigenvalue zero equals k. Its associated eigenspace has a basis $\gamma_1, \gamma_2, \ldots, \gamma_k$ in $\mathbb{R}^N$ whose elements γ_i satisfy: $(i)\,\gamma_i(v)=0$ for $v \in G - R_i$; $(ii)\,\gamma_i(v)=1$ for $v \in H_i$; $(iii)\,\gamma_i(v) \in (0,1)$ for $v \in C_i$; $(iv)\,\sum_i \gamma_i = \mathbf{1}_N$.

Proof. The first part of the statement is a straightforward application of Gershgorin's Theorem (see ref. 2). The second and third are extensions (proved in ref. 3) of the well-known theorem that asserts that the Laplacian of a connected undirected loopless graph has a unique zero eigenvalue (for example, see ref. 2 or 7) whose associated eigenvector has constant entries. ∎

Remark. One can in fact easily solve for $\gamma_i|_{C_i}$ and give an explicit formula for it. Since this would involve some definitions we refer the reader to ref. 3.

Remark. If the number of reaches in the graph is one, the graph has a spanning tree, and conditions (i) through (iv) reduce to the case where the nullspace is spanned by the all ones vector.

4. FORMATION AND STABILITY

In this section we analyze the behavior of the solutions of the stabilized (see Definition 4.3) system. The main result (Theorem 4.4) is that K can be chosen so that we can identify a set $\mathcal{W}$ of solutions associated with the zero eigenvalue of the Laplacian and such that every solution is asymptotic to a solution in $\mathcal{W}$ (stability is global since Equation 4.3 is linear). The set $\mathcal{W}$ is characterized in terms of the connectivity properties of the underlying communication graph. In particular if the graph has precisely k reaches then the set $\mathcal{W}$ has dimension $2dk$: for each reach, one can specify a position and velocity.

The separation of the momentum from the position parts of any vector is done for physical reasons: We want to maintain the relation $\dot{x}_k = v_k$ for each agent. This leads to the following specifications: For each agent we have

$$\rho_i = x_i \otimes e_1 + v_i \otimes e_2 \quad \text{where} \quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Similarly, for any $2d$ by $2d$ matrix C

$$C = \sum_{i=1}^{4} C_i \otimes J_i, \tag{4.1}$$

where the matrices J_i are defined by:

$$J_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad J_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The meaning of this decomposition is that C_1 pairs position-like entries with position-like entries, C_2 pairs position with velocity, and so on. We will use this decomposition whenever necessary, so that A and K for example have decompositions in terms A_i and K_i, respectively.

If we want to maintain the relation $\dot{x}_k = v_k$ together with equation 2.3 it is easy to see that we must require the following.

Remark 4.1. For physical reasons we define $A_1 = 0$, $A_2 = I_d$, and $K_1 = K_2 = 0$.

To formulate the next few results, it will be convenient to define new coordinates z. Define

$$\begin{aligned} &y = z - h - \mathbf{1}_N \otimes \alpha \quad \text{or} \\ &z = y + h + \mathbf{1}_N \otimes \alpha = y + h + \mathbf{1}_N \otimes \left(q \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + p \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right). \end{aligned} \tag{4.2}$$

In the following statement we show, among others, that if we allow h to have position AND velocity-like components, we can without loss of generality choose the velocity-like components to be zero. So for the purpose of the following proposition alone set

$$h = \sum_{k=1}^{N} e_k \otimes \left(\xi_k \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \eta_k \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right).$$

Proposition 4.2. Given N agents equipped with a fixed communication graph. Then

i): The system given by Equation 2.3 admits an in formation solution for every desired constant configuration $\{\xi_k\}_{k=1}^N$ *if and only if* $A_3 = 0$ *and* (without loss of generality) all the η_k are 0.

ii): In formation, the position and velocity of the system as a whole satisfy

$$\dot{q} = p$$
$$\dot{p} = A_4 p$$

Proof. We start by proving the 'only if' part of item i). Use the above coordinate transformation to eliminate z. Clearly, if the system is in formation then $y=0$. Upon setting $y=0$ and using that $L_N \cdot \mathbf{1}=0$, Equation 2.3 becomes:

$$\dot{h} + \mathbf{1}_N \otimes \dot{\alpha} = \mathbf{1}_N \otimes (A\alpha) + I_N \otimes Ah + 0.$$

Using the above notation together with the hypothesis that $\dot{h}=0$, we get

$$\mathbf{1} \otimes \left[\dot{q} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \dot{p} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \mathbf{1} \otimes \left[\left(\sum_{i=1}^{4} A_i \otimes J_i\right)\left(q \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + p \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)\right]$$
$$+ \sum_{k=1}^{N} e_k \otimes \left[\left(\sum_{i=1}^{4} A_i \otimes J_i\right)\left(\xi_k \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \eta_k \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)\right]$$

From this it is clear that the terms in the last set of square brackets do not depend on the index k. This is only possible if for all k, the $2d$-vector $(\xi_k - \xi_1) \otimes e_1$ is in $\mathrm{Ker}(\sum_{i=1}^{4} A_i \otimes J_i)$, and by assumption, this can only happen if $A_3 = 0$. On the other hand, recall that, for physical reasons, we assumed that $A_2 = I_d$ (see Remark 4.1). Then for the same reason we must have that the differences of the η_k must all be zero. Thus (for all k) we set η_k equal to η_0. Without loss of generality, the resulting term can be absorbed in the p-term of the equation. Now notice that the last term (the one containing e_k) is zero and the 'only if' part of i) follows.

The 'if' part of the first statement and the second statement of the theorem follow directly from writing out the resulting equations. ∎

Remark. In the case that $A_4 \neq 0$, the equations do not respect Galilean invariance; the agents observe their *velocity with respect to the ground* and accelerate accordingly. Also if the diagonal entries of A_4 are all equal, the agents do not change direction while they accelerate. (In that case the acceleration depends only on the ground*speed*.) We will assume from now on that all η_k are zero. Note, though, that the a priori given off-set h needs to be evaluated in the original coordinate system (see however Section 8).

In the following theorem we assume that we are given N agents equipped with a fixed communication graph with k reaches. Use the notation $\{\gamma_i\}_{i\in\{0,\dots k\}}$ for a fixed set of eigenvectors associated with the zero eigenvalue of the Laplacian of the communication graph (see Theorem 3.6). Let $\{\lambda_i\}_{i=k+1}^{N}$ be the non-zero eigenvalues of L_N.

Definition 4.3. The system given by Equation 2.3 is called *stabilizable* if K can be chosen such that for each non-zero λ in the spectrum of L, all eigenvalues of $A+\lambda K$ have real part strictly less than zero (and *stabilized* if K has been chosen this way).

Theorem 4.4. Suppose the system given in Equation 2.3 is stabilized. Then every orbit is asymptotic to an orbit in $h+\mathcal{V}$ where $\mathcal{V}$ is the space of linear combinations of $\{\gamma_i\otimes\rho_j\}_{i\in\{0,\dots k\},j\in\{1,\dots 2d\}}$, where the γ_i span $\mathrm{Ker}(L)$ and are as in Theorem 3.6 and the ρ_j are $2d$ independent solutions of $\dot\rho_j=A\rho_j$ as in Proposition 4.3.

Remark. Recall that the vectors γ_i can be explicitly calculated from the Laplacian.

Proof. By Proposition 4.2 we know that in formation solutions exist. So, choose α so that $\dot\alpha=A\alpha$. After the coordinate change given in Equation 4.2, Equation 2.3 becomes a $2dN$-dimensional autonomous ordinary linear differential equation of the form:

$$\dot y = I_N\otimes Ay + L_N\otimes Ky \equiv My. \tag{4.3}$$

The solutions $e^{Mt}y_0$ of this system form a $2dN$-dimensional linear space. As is well-known the components of any solution $y(t)$ is a sum of terms of the form (see ref. 1): $p(t)e^{\mu t}$, where μ is an eigenvalue of M and p is a polynomial in t whose degree is less than the geometric multiplicity of that eigenvalue.

Next observe that I_N and L_N are simultaneously triangularized by conjugating M with $U\otimes I_{2d}$ where U brings L into upper triangular form (see also ref. 12). One easily verifies that

$$(U^{-1}\otimes I_{2d})(I\otimes A+L\otimes K)(U\otimes I_{2d})$$

is block upper triangular and that its diagonal blocks are of the form $A+\lambda K$, for λ an eigenvalue of L. The hypothesis in the Theorem now implies that if $\phi(t)$ is a solution that can be written as a linear combination of exponentials involving *only* eigenvalues of $A+\lambda K$ for λ non-zero, then it is a linear combination of decaying eigen solutions.

On the other hand, assume that $\phi(t) = \gamma_i \otimes \rho_j$ and substitute into Equation 4.3. One obtains:

$$\gamma_i \otimes \dot{\rho}_j = \gamma_i \otimes (A\rho_j).$$

In accordance with Proposition 4.2, this gives us $2d$ linearly independent solutions for each i. ▮

Definition 4.5. A set $\mathcal{W}$ of solutions $\phi : \mathbb{R} \to X$ is called *locally stable* if every solution starting nearby is asymptotic to a solution in $\mathcal{W}$. The set $\mathcal{W}$ is *globally stable* if this is true for every initial condition. $\mathcal{W}$ is *minimal* and stable if it has no proper subset satisfying the same (local) condition. (In the non-autonomous case, consider sets of solutions $\phi : \mathbb{R} \to X \times R$ in the extended phase-space.)

Note that the standard notions of "attractor" do not easily apply in this case due to the lack of compactness of the involved spaces. (For a detailed discussion of these and related concepts, see ref. 14).

With the above definition we can thus summarize the main result of this section (Theorem 4.4) as follows: if the system is stabilized, then we can identify a stable set associated with the zero eigenvalue of the Laplacian of the communication graph. The proof that (for a large set of constant matrices A) a system given by Equation 2.3 can be stabilized will be taken up in the next section.

As an example of the ideas in this section, consider a system of 5 agents on the line ($\mathbb{R}$). In this case the matrix A_4 is a scalar a. Suppose also that h is given by

$$h = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and furthermore that its Laplacian L is the one given in the example at the end of Section 3. Since this graph has two reaches, the eigenspace corresponding to the zero eigenvalue is two-dimensional. It is easy to verify that it is spanned by

$$\gamma_1 = \begin{pmatrix} 1 \\ 1 \\ 2/3 \\ 1/3 \\ 0 \end{pmatrix} \quad \text{and} \quad \gamma_2 = \begin{pmatrix} 0 \\ 0 \\ 1/3 \\ 2/3 \\ 1 \end{pmatrix}.$$

Thus if $a \neq 0$ the set $\mathcal{V}$ of linear combinations of $\{\gamma_1, \gamma_2\}$ Kronecker-multiplied with vectors of the form

$$\left\{ \left(q_0 - \frac{p_0}{a} \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{p_0}{a} \begin{pmatrix} 1 \\ a \end{pmatrix} e^{at} \right\}$$

form a 4-dimensional stable linear space of solutions to Equation 4.3 with initial condition $\rho(0) = (q_0, p_0)$. (If $a = 0$, the matrix A is a Jordan matrix of order 2 with associated eigenvalue 0, and we get a degree one polynomial in the second term.) The set $h + \mathcal{V}$ is a stable set for Equation 2.3. It is the unique minimal stable set only if $a \geqslant 0$. If $a < 0$ the (unique) minimal stable set is given by $h + \mathcal{V}$ together with the condition that $p_0 = 0$.

With this notation we see that the last theorem reduces to specifying a (globally) stable set. Note that generally this is a set much bigger than the set of in formation solutions. However, there is a special case discussed earlier in the literature (see Introduction) which immediately follows from this.

Corollary 4.6. Suppose the system given in 2.3 is stabilized. Then the set of in formation solutions is globally stable if and only if its communication graph of G has exactly one reach.

For instance a (loopless) connected undirected graph always has one reach, and thus the above corollary applies to it (see also ref. 12). Again, minimality applies in a more restricted set of circumstances.

5. THE PROOF OF STABILIZABILITY AND APPLICATIONS

The matrix A in Equation 2.3 can be used to steer the flock according to Proposition 4.2 (provided we can stabilize the system). In this section we give the proof that for various important choices of A, the system can indeed be stabilized by choosing K_3 and K_4 appropriately. The freedom of choice is sufficient to linearly accelerate the formation or steer it along circular arcs, thus permitting great freedom in the advance programming of the orbit. We show this by choosing the entries of K_3 and K_4 appropriately and showing that Equation 2.3 is stabilized and then apply Theorem 4.4.

In this section we will use the following notation:

$$\epsilon \equiv \min_{\lambda \in \mathrm{spec}(L)-\{0\}} \{\mathrm{Re}(\lambda)\} > 0.$$

We will also decompose matrices as before (see Equation 4.1). Denote the diagonal elements of A_4 by $a_m \in \mathbb{R}$, those of K_3 by $f_m \in \mathbb{R}$ and those of K_4 by $g_m \in \mathbb{R}$ (where m ranges between 1 and d). In the following proposition, the orbit is steered by the matrix in accordance with Proposition 4.2; we prove that there is a filter K such that the flock flies in formation for a large set of matrices A. (The matrix A_4 is time independent. If we allow it to be time dependent, stability is a much more complicated issue.) A more general statement is given in the main result of this section, Theorem 5.2.

Proposition 5.1. Suppose A_4, K_3, and K_4 are diagonal. Given A_4, the system can always be stabilized by setting (g_k, f_k) so that for all $k \in \{1, \ldots, d\}$:

$$\begin{array}{c} f_k < 0 \\ g_k < 0 \\ f_k > -g_k(\epsilon g_k + a_k) \end{array}.$$

Proof. First, notice that the conditions of the theorem are equivalent with the statement that:

$$\begin{array}{c} f_k < 0 \\ g_k < 0 \\ \epsilon > \max\left\{-\frac{f_k + a g_k}{g_k^2}, 0\right\}. \end{array} \tag{5.1}$$

It follows from the proof of Theorem 4.4 that the fundamental modes of Equation 4.3 not associated with the zero eigenvalue of the Laplacian decay as $e^{\nu t}$, where ν is the real part of the eigenvalues of $A + \lambda K$ for $\lambda \in \mathrm{spec}(L) \setminus \{0\}$. According to this and Remark 4.1, we obtain for the stability calculation:

$$\begin{aligned} A + \lambda_i K &= I_d \otimes J_2 + \lambda_i K_3 \otimes J_3 + (A_4 + \lambda_i K_4) \otimes J_4 \\ &= \sum_{k=1}^{d} E_k \otimes \begin{pmatrix} 0 & 1 \\ \lambda_i f_k & a_k + \lambda_i g_k \end{pmatrix}, \end{aligned}$$

where E_k is the $d \times d$ matrix whose only non-zero entry is 1 at the k-th location on the diagonal. For brevity we now drop the subscript k for the course of this proof. We need to make sure that for any given a, we can choose f and g such that for all $\lambda \in \text{spec}(L) - 0 \subset (\{1\} + B_1) \setminus \{0\}$, the roots $x_\pm$ of

$$x^2 - (a + \lambda g)x - \lambda f = 0 \tag{5.2}$$

have negative real part.

Our argument relies on continuity. Since

$$x_\pm = a + \lambda g \pm \sqrt{(a + \lambda g)^2 + 4\lambda f},$$

we see (using Equation 5.1) that if $\lambda = \lambda_0 \in \mathbb{R}^+$ and very large then both roots have negative real parts. Suppose that for some λ_s in the spectrum of L one of the roots $x_\pm$ of the above characteristic polynomial has positive real part. Then (by the Intermediate Value Theorem) there must be a value λ on the segment connecting λ_0 and λ_s such that one of the roots has real part zero. Upon substitution of $x_\pm = i\tau$ where τ is real, into that polynomial, one easily sees that:

$$\begin{aligned}\exists\, \tau \in \mathbb{R} \text{ such that } \lambda &= \frac{-\tau(\tau + ia)}{(f + ig\tau)} \\ &= -\frac{(f + ag)}{g^2} \frac{g^2\tau^2}{f^2 + g^2\tau^2} + i(\text{Imaginary Part}).\end{aligned}$$

Since by construction $\text{Re}(\lambda) > \text{Re}(\lambda_s) \geqslant \epsilon$, this contradicts Equation 5.1. Thus for any λ_s in the spectrum of L (except 0) the associated eigenvalues have negative real parts. ∎

Remark. The above criterion is sufficient but not necessary. It could for instance be improved by using the fact that the imaginary part of λ has modulus less than or equal to 1. It is in fact possible to get a necessary and sufficient criterion (as was done in ref. 12): Multiply the above characteristic polynomial with its conjugate and then use the Routh-Hurwitz Theorem which gives necessary and sufficient conditions for a polynomial with real coefficients to have roots with negative real parts. However, our current method has the advantage of giving very concise sufficient criteria.

Remark. An alternative approach yields an optimal result. The price one pays is that the resulting criterion is very unwieldy. We outline it here. It is easily checked that Equation 5.2 with $a = 0$ has roots with negative real parts if and only if $f<0$ and $g<0$. Set $x=i\tau$ and $\lambda=\lambda_r+i\lambda_i$. When $a\neq 0$, that equation has roots with negative real parts if and only if

$$a=-\frac{\lambda_i f}{\tau}-\lambda_r g+i\left(\frac{\lambda_r f}{\tau}-\lambda_i g+\tau\right).$$

Since a is real, the imaginary part must be zero. This yields a quadratic equation in τ. If that equation has real roots $\tau_\pm$, define $I(\tau_\pm)$ be the maximal open interval containing zero and not containing $-\frac{\lambda_i f}{\tau_\pm}-\lambda_r g$. In this case we get a condition of the form: If a is in the intersection of the intervals $I(\tau_\pm)$, then Equation 5.2 has roots with negative real part. Furthermore, of all intervals containing zero, this intersection is the maximal interval with that property.

From the above we see that the matrix A_4 can be used to affect velocity changes. If the system at time $t=0$ is supposed to have 'average' velocity p_0 and p_1 at time $t=1$, then one has

$$p_1=\exp(A_4)\,p_0.$$

Since A_4 is assumed to be diagonal its elements can easily be calculated. There are two problems with this. First, for diagonal A_4, the above equation implies that the components of the average velocity cannot change sign. Second, this acceleration is exponential and so for any physical object can be maintained only for short times. The following result indicates how in addition to the above, in formation maneuvers can be made that change the direction of the flock's motion into any arbitrary other direction. Note that it is sufficient to prove that one can describe circles in a plane. Without loss of generality we may thus prove this theorem for $d=2$. In the Theorem, set $K_3=fI_2$ and $K_4=gI_2$.

Theorem 5.2. Fix some $a_0>0$. Assume that K is as in Proposition 5.1 with a_k replaced by a_0. Then

i): If A_4 is diagonal and its diagonal entries a_k satisfy $|a_k|<a_0$, then each component stably accelerates (or decelerates) according to $v(t)=v(0)\,e^{at}$ where $|a|\leqslant a_0$, or

ii): If $A_4=\begin{pmatrix}0 & -k\\ k & 0\end{pmatrix}$ and $|k|\leqslant 2\sqrt{|1+\epsilon\frac{g^2}{f}|\cdot|\epsilon f|}\neq 0$, then the formation stably describes circles at constant speed $v_0\neq 0$, with curvature $\kappa=k/v_0$.

Proof. The first part of the theorem is of course the same as the previous proposition.

We may assume that $v_0 \neq 0$. The conditions of Proposition 5.1 imply that $\sqrt{|1+\epsilon\frac{g^2}{f}| \cdot |\epsilon f|} \neq 0$.

We begin by recalling that a curve $z(t)$ in the complex plane parametrized by arc-length satisfies:

$$\ddot{z} = ik(t)\dot{z}.$$

Here k is a real function and its absolute value is equal to the curvature. (In fact the curve is uniquely determined by $k(t)$ and an initial condition $(z(0), \dot{z}(0))$.) If the curve is such that (as in our case) $|\dot{z}(t)| = v_0$, then the curvature equals $|k|/v_0$. We have chosen the convention that $\ddot{z}$ makes an angle of $+\pi/2$ with the tangent-vector $\dot{z}$ to the curve. Of course, we take k to be constant (to avoid having to deal with a non-autonomous system of equations). Now call the real direction x and the imaginary direction y and call the components of a tangent vector (v, w). Then we can translate this as follows (using 2 spatial and 2 velocity components):

$$\frac{d}{dt}\begin{pmatrix} x \\ v \\ y \\ w \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -k \\ 0 & 0 & 0 & 1 \\ 0 & k & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \\ y \\ w \end{pmatrix}.$$

This corresponds to setting

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= A_4 p \end{aligned} \quad \text{where} \quad A_4 = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}.$$

We obtain as before that

$$A + \lambda_i K = I_d \otimes J_2 + \lambda_i K_3 \otimes J_3 + (A_4 + \lambda_i K_4) \otimes J_4.$$

But now A_4 is not diagonal. Recalling the assumptions on K_3 and K_4 and denoting the eigenvalue λ_i by λ, it is easy to see that

$$A + \lambda K = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \lambda f & \lambda g & 0 & -k \\ 0 & 0 & 0 & 1 \\ 0 & k & \lambda f & \lambda g \end{pmatrix}.$$

It follows upon inspection of $A + \lambda K$ that the eigenvalues x of $A + \lambda K$ are determined by the zeros of the following polynomial equation:

$$k^2 x^2 + (x^2 - \lambda g x - \lambda f)^2 = 0. \tag{5.3}$$

Now if $k=0$, Proposition 5.1 affirms that all roots have negative real part.

As before, the argument relies on the Intermediate Value Theorem. The second statement of the theorem is false if and only if there is an intermediate value for k such that $x_\pm = i\tau$ (where $\tau \in \mathbb{R}$) is a solution of the above equation. Now $\tau = 0$ together with the conditions of Proposition 5.1 lead to a contradiction. So, since $\tau \neq 0$, the above equation now becomes

$$k^2 = \left(\tau + i\lambda g + \frac{\lambda f}{\tau}\right)^2.$$

Set $\lambda = \lambda_r + i\lambda_i$. Since k is real, we must have that $\tau + i\lambda g + \frac{\lambda f}{\tau}$ is real. This gives

$$\lambda_i = -\frac{g\tau}{f}\lambda_r, \tag{5.4}$$

and so:

$$k = \pm\left(\left(1 + \frac{g^2\lambda_r}{f}\right)\tau + \lambda_r f \frac{1}{\tau}\right). \tag{5.5}$$

Using the inequalities in the conditions of Proposition 5.1 (note that $f + \epsilon g^2 > -a_0 g = 0$) and the fact that $\lambda_r \in [\epsilon, 2]$, one sees that both coefficients of this rational function of τ are negative. It is then a straightforward calculation to show that the minimum of $|k|$ as a function of τ is twice the square root of the product of these coefficients. Thus

$$\text{if} \quad \forall \lambda \in \operatorname{spec}(L) - \{0\} : |k| \leqslant 2\sqrt{|1 + \operatorname{Re}(\lambda)\frac{g^2}{f}| \cdot |\operatorname{Re}(\lambda) f|} \neq 0,$$

then for those values of λ equation 5.3 cannot have a root. This yields the estimate for k. The theorem follows by noting that the curvature κ is given by the value of k divided by the velocity of the flock. ∎

Remark. The configuration of the flock during the manoeuver described in the second part of the theorem does not change. Each agent in the flock follows a translate of the same path. Thus the formation itself does not change its orientation.

Remark. One may get a stabler system if one allows the matrices K_3 and K_4 to have non-zero off-diagonal elements as well.

Remark. One can use Equation 5.4 to eliminate τ from Equation 5.5. This immediately leads to a sharper (if less concise) criterion. The formation stably describes circles if and only if for all $\lambda = \lambda_r + i\lambda_i \in \mathrm{spec}(L) - \{0\}$:

$$|k| \leqslant -\frac{f}{g}\frac{\lambda_i}{\lambda_r} - g\lambda_i - g\frac{\lambda_r^2}{\lambda_i}$$

This is derived by the Routh-Hurwitz method in ref. 28.

Let us return to the example at the end of Section 4. Since it deals with agents on line, we cannot make turns. However, Theorem 5.1 still applies. Now it is again straightforward to convince oneself that the characteristic polynomial of its Laplacian (given at the end of Section 3) is

$$\xi_L(x) = -x^2\left(x - \frac{1}{2}\right)(x-1)\left(x - \frac{3}{2}\right).$$

For the formation to remain intact at accelerations less than or equal to 1 (in absolute value), we must choose the filter K such that

$$K = \begin{pmatrix} 0 & 0 \\ f & g \end{pmatrix} \quad \text{where} \quad \begin{array}{l} f < 0 \\ g < 0 \\ f > -g(\frac{g}{2}+1) \end{array}.$$

For example, $f = -1$ and $g = -4$ works.

6. FLOCKS WITH INDEPENDENT LEADERS

In this section we analyze the behavior of a (stabilized) flock as it tries to follow one or more leaders whose orbits are prescribed functions of time (independent of the orbits of the other agents). Such leaders are called independent. The theory in the previous sections allows for the presence only of dependent leaders (see below). Here we extend that theory in the sense that we assume there is at least one independent leader.

The equations of motion for the position x_ℓ and velocity v_ℓ of a leader (see Definition 3.1) at vertex ℓ are given by (compare also with the explicit equations at the end of Section 2)

$$\dot{x}_\ell = v_\ell \quad \text{and} \quad \dot{v}_\ell = A_4 v_\ell. \tag{6.1}$$

Recall now that each reach of the communication graph has at most one leader. We will now assume that at least one of the leaders will follow an a priori given orbit instead of its orbit being determined by Equation 6.1 plus an initial condition. More precisely:

Definition 6.1. An agent at ℓ is called an *independent leader* if it is a leader and if, instead of by Equation 6.1, its orbit is determined by an a priori given differentiable function $\psi_\ell : \mathbb{R} \to \mathbb{R}^d$:

$$x_\ell \equiv \psi_\ell(t) \quad \text{and} \quad v_\ell \equiv \dot{\psi}_\ell(t) \quad \text{or} \quad z_\ell(t) \equiv \psi_\ell(t) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \dot{\psi}_\ell(t) \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A leader that is not independent will also be called a dependent leader and its orbit satisfies Equation 6.1.

Assume we have $N+k$ agents of which k are independent leaders. The vertices representing independent leaders will be denoted by $\mathcal{L}$. Without loss of generality we can assume that the desired position for an independent leader is zero: for $\ell \in \mathcal{L}$ we have $h_\ell = 0$. The vertices not in $\mathcal{L}$ will be denoted by 1 through N. Remove from the Laplacian L_{N+k} those rows that correspond to independent leaders and in the resulting N by $N+k$ matrix write the column corresponding to the i-th vertex of this matrix as L_i. (These columns are considered as vectors below.) Now denote by $P = P_N$ the "reduced (directed) Laplacian" obtained by subsequently also removing the columns that correspond to the independent leaders. Thus the i-th column of P_N corresponds to the vertex i which by assumption is not an independent leader. With a slight abuse of notation we will again use z for the position-velocity vector of agents 1 through N. With this notation, it is straightforward to verify the following lemmas (see also the example that follows the lemmas).

Lemma 6.2. With the above notation ($\mathcal{L}$ the set of independent leaders), the equations of motion become:

$$\dot{z} = I_N \otimes Az + P_N \otimes K(z-h) + \sum_{\ell \in \mathcal{L}} L_\ell \otimes (Kz_\ell(t)). \tag{6.2}$$

Performing the coordinate change $y = z - h$ again, and making some obvious abbreviations, the above equation of motion reads:

$$\dot{y} = (I_N \otimes A + P_N \otimes K)y + g(t) = My + g(t). \tag{6.3}$$

Lemma 6.3. The characteristic polynomials of L_{N+k} and P_N satisfy: $x^k \chi_P(x) = \chi_L(x)$ where $k = |\mathcal{L}|$ is the number of independent leaders.

Let us illustrate these ideas with our (by now) standard example. In it, rows (or columns) 1 and 5 correspond to leaders. Let us assume that both are independent leaders and that their orbits are determined by the functions ψ_1 and ψ_5 respectively. The equations of motion become (after relabeling rows 2, 3, and 4 of the old Laplacian L_5, to become rows 1, 2, and 3 of the reduced Laplacian P_3):

$$\dot{z} = I_3 \otimes \begin{pmatrix} 0 & 1 \\ 0 & a \end{pmatrix} z + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{pmatrix} \otimes K(z-h) + \begin{pmatrix} -1 \\ -\frac{1}{2} \\ 0 \end{pmatrix} \otimes \left(K \begin{pmatrix} \psi_1 \\ \dot{\psi}_1 \end{pmatrix} \right)$$
$$+ \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2} \end{pmatrix} \otimes \left(K \begin{pmatrix} \psi_5 \\ \dot{\psi}_5 \end{pmatrix} \right).$$

Note that z and h now have 3 position-like and 3 velocity-like coordinates. In this example every original reach has a leader which is now independent. Observe that P has no zero eigenvalue. Since we can choose K such that the original system is stabilized, and by the last lemma, P_3's eigenvalues are identical to the nonzero ones of L_5, we expect $I_3 \otimes A + P_3 \otimes K$ to have negative eigenvalues for the same choice of K (discussed at the end of Section 5). A stable set of solutions is thus given by the "particular solution" to the non-homogeneous equation (all other solutions are asymptotic to it).

This example illustrates the general principle we state in the next theorem. We assume that we are given $N+k$ agents, of which k are independent leaders denoted by $\mathcal{L}$, equipped with a fixed directed loopless communication graph G. Let $G = \cup_{i=1}^{k} R_i$ be the unique union of reaches of G (Definition 3.5). Index the reaches and let $\mathcal{I}$ denote the set of indices for which the following is true:

$$i \in \mathcal{I} \quad \Leftrightarrow \quad R_i \text{ contains no independent leader.}$$

Use the notation $\{\gamma_i\}_{i\in\mathcal{I}}$ for a fixed set of eigenvectors associated with the zero eigenvalue of the Laplacian of the communication graph (see Theorem 3.6).

Theorem 6.4. Suppose K is chosen such that the original Laplacian system is stabilized. Let $y_p(t)$ be any particular solution of Equation 6.2. Then every orbit is asymptotic to an orbit in $h + y_p(t) + \mathcal{V}$ where $\mathcal{V}$ is the space of linear combinations of $\{\gamma_i \otimes \rho_j\}_{i\in\mathcal{I}, j\in\{1,\ldots,2d\}}$, where the γ_i are in the span of $\mathrm{Ker}(L_{N+k})$ and are as in Theorem 3.6 and the ρ_j are $2d$ independent solutions of $\dot{\rho}_j = A\rho_j$ as in Proposition 4.2.

Remarks. We point out that the set $h + y_p(t) + \mathcal{V}$ identified in the theorem is a minimal (globally) stable set. Note also that if $\mathcal{I}$ is empty, we have the situation of the example, and all solutions will be asymptotic to the sum of $h + y_p(t)$.

Proof. Do the substitution to obtain Equation 6.3. The general solution of that equation is given by (see refs. 1, 9)

$$y(t) = e^{Mt} y(0) + e^{Mt} \int_0^t e^{-Ms} g(s) ds.$$

The second term of the right hand side is $y_p(t)$, the particular solution of the theorem. Since every solution of Equation 6.3 can be written as the sum of the particular solution $y_p(t)$ and a solution of the homogeneous equation

$$\dot{y} = (I_N \otimes A + P_N \otimes K)y = My,$$

the problem is now reduced to finding the asymptotic orbits of the homogeneous equation. Keeping in mind the relation between the spectra of L and P (Lemma 6.3), the proof of Theorem 4.4 now applies verbatim. ∎

We have not mentioned in formation solutions yet. The reason is that one does not in general expect the stable set we described in the last theorem to contain in formation solutions. To see this, we will take the analysis of our earlier example a bit further.

In order for there to be in formation solutions, certainly the positions of the two leaders must maintain a constant difference. Thus, let us set $\psi_1(t) = \psi_5(t) = k\cos(\omega t)$. To ensure stability at least against constant acceleration, we follow Section 5 and set $f = -1$ and $g = -4$ in the linear feedback K. Since the aim is to follow the leader we also set $A_4 = a = 0$. Going back to Equation 6.3 we can write:

$$g(t) = \begin{pmatrix} -1 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \otimes \left(\begin{pmatrix} 0 & 0 \\ f & g \end{pmatrix} \begin{pmatrix} k\cos(\omega t) \\ \omega k \sin(\omega t) \end{pmatrix} \right) = \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} 0 \\ c_\omega \end{pmatrix} \cos(\omega t - \phi_\omega).$$

Here we have set

$$c_\omega = k\sqrt{g^2\omega^2 + f^2} = k\sqrt{16\omega^2 + 1} \quad \text{and} \quad \phi_\omega = \arctan\left(\frac{g\omega}{f}\right) = \arctan(4\omega).$$

Using a new time-variable τ such that $\omega\tau = \omega t - \phi_\omega$, Equation 6.3 becomes the real part of this equation:

$$\dot{y} = My + \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} 0 \\ c_\omega \end{pmatrix} e^{i\omega\tau} = My + g_0\, e^{i\omega\tau},$$

where, of course, the dot represents differentiation with respect to τ and g_0 is a 6-dimensional vector. One can derive (variation of constants and the Ansatz that $y = \zeta e^{i\omega\tau}$ see ref. 1 or 9)—or simply check by substitution—that the general complex valued solution of the equation is

$$y_c(\tau) = e^{M\tau} y_0 + (M - i\omega I_3)^{-1}(e^{M\tau} - e^{i\omega\tau} I_3)\, g_0.$$

We are interested in the real part of an asymptotic solution. Because we have chosen K such that spec(M) is strictly contained in the negative left half of the complex plane, we can write such a solution as:

$$\begin{aligned} y(\tau) &= \Re\left\{-(M - i\omega I_3)^{-1} e^{i\omega\tau}\right\} g_0 \\ &= -(M^2 + \omega^2)^{-1}(M\cos(\omega t - \phi_\omega) - \omega I_3 \sin(\omega t - \phi_\omega))\, g_0. \end{aligned}$$

If ω is very small then the sine-term and ϕ_ω both are negligible and the phase-shift will be small. However, if ω is big, the sine-term dominates and ϕ_ω is large and phase-shifts appear. This conclusion is independent of the graph and even of the values of f and g as long as they satisfy the conditions in Section 5. Thus we expect phase-shifts in general. We have not thoroughly investigated the amplitudes here. We only have established that in general the flight pattern will not be in formation.

Remark 6.5. A flock with an independent leader whose position is a (non-constant) function of time, will not in general be asymptotic to an in formation solution.

We finally remark that this last calculation is reminiscent of the elementary treatment of the non-homogeneous heat-equation (see ref. 8 for example). This is no coincidence. In some sense these systems (if stabilized) correspond to a heat equation with the spatial variable discretized.

7. HIERARCHICAL FLOCKS

In this section we will show how large communication graphs can be assembled from smaller ones in such a way that the calculation of the stability is not affected by assembling the graphs. The obvious advantage of this construction is that for certain very large graphs, the stability calculation in Theorem 4.4 involves only a low-dimensional Laplacian and the $2d$-dimensional matrices A and K, and therefore is greatly simplified. This leads to the construction of hierarchical graphs G_n and their associated Laplacians L_n which appeared in ref. 27. We give a simplified account of this construction and extend the results.

Definition 7.1. Let g_1 and g_2 be two directed graphs. The connected sum $g = g_1 \cup_{i_1=i_2} g_2$ is the graph obtained by attaching g_2 to g_1 through the identification the vertices $i_1 \in g_1$ and $i_2 \in g_2$.

Denote by ℓ_i the Laplacian associated to g_i and by ℓ the Laplacian of the union $g = g_1 \cup_{i_1=i_2} g_2$. The characteristic polynomial of a Laplacian L is written as $\chi_L(x)$.

Lemma 7.2. If $i_2 \in g_2$ is a leader then the characteristic polynomial of the Laplacian of the union $g = g_1 \cup_{i_1=i_2} g_2$ is given by $\chi_\ell(x) = \chi_{\ell_1}(x) \left(\frac{\chi_{\ell_2}(x)}{x}\right)$.

Proof. First arrange the Laplacian matrices ℓ_i in such a way that i_1 is the last row of ℓ_1 and i_2 the first of ℓ_2. The proof of this observation follows most conveniently by inspection of the Laplacian matrix ℓ. ∎

Definition 7.3. Let g_1 and g_2 be two directed graphs with n_1 and n_2 vertices each. The *hierarchical product* $g = g_1 \times_{i_2} g_2$ is the graph obtained by attaching a copy of g_2 to each vertex i of g_1 (every time by identifying the vertices $i \in g_1$ and $i_2 \in g_2$).

By repeating the connected sum construction we thus immediately obtain the following results for this hierarchical product.

Proposition 7.4. (see ref. 27) Suppose i_2 is a leader. The Laplacian ℓ of $g = g_1 \times_{i_2} g_2$ is given by $\ell = \ell_1 \otimes E_{n_2,1} + I_{n_1} \otimes \ell_2$ where $E_{N,1}$ is the $N \times N$ matrix whose only non-zero entry is a leading one, and I_N stands for the $N \times N$ identity matrix.

Proposition 7.5. Suppose i_2 is a leader. The Laplacian ℓ of $g = g_1 \times_{i_2} g_2$ has characteristic polynomial $\chi_\ell(x) = \chi_{\ell_1}(x) \left(\frac{\chi_{\ell_2}(x)}{x}\right)^{n_1}$.

To aid in the understanding of the notation, we consider a simple example. Suppose g_1 is a 2-vertex graph with Laplacian ℓ_1, and g_2 is a 3-vertex graph with Laplacian ℓ_2, where:

$$\ell_1 = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad \ell_2 = \begin{pmatrix} 0 & 0 & 0 \\ -1/2 & 1 & -1/2 \\ 0 & -1 & 1 \end{pmatrix}.$$

Both graphs have the property that their first entry corresponds to the leader. Now define $g_3 = g_1 \times g_2$. Its associated Laplacian ℓ_3 equals $\ell_1 \otimes E_{3,1} + I_2 \otimes \ell_2$. Thus

$$\ell_3 = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ -1/2 & 1 & -1/2 \\ 0 & -1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1/2 & 1 & 1/2 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

It is easy to see that the characteristic polynomial of ℓ_3, $\chi_{\ell_3}(x)$, is given by

$$\chi_{\ell_3}(x) = \chi_{\ell_1}(x) \left(\frac{\chi_{\ell_2}(x)}{x} \right)^2.$$

Thus the eigenvalues in the spectrum of ℓ_3 is determined by those of ℓ_1 and ℓ_2 only. The process can be continued by defining for example $g_4 = g_2 \times g_3$. For the characteristic polynomial of the associated Laplacian we obtain

$$\chi_{\ell_4}(x) = \chi_{\ell_2}(x) \left(\frac{1}{x} \cdot \frac{\chi_{\ell_1}(x)(\chi_{\ell_2}(x))^2}{x^2} \right)^3 = \frac{(\chi_{\ell_1}(x))^3 (\chi_{\ell_2}(x))^7}{x^9}.$$

It is clear from this example that by starting with a few Laplacians with a known spectrum, we can obtain ever larger graphs by making the above procedure recursive. More generally we can use unions and products to obtain ever larger graphs without changing the eigenvalues of the spectrum of the associated new Laplacian, as long as the second graph in

$g_{n+1} = g_n \cup_{j_n=j_{n-1}} g_{n-1}$ or $g_{n+1} = g_n \times_{j_{n-1}} g_{n-1}$ is attached to the first by way of a leader (j_{n-1}). Let us call the associated Laplacians ℓ_n and ℓ_{n-1}. We thus obtain that the eigenvalues of the Laplacian ℓ_{n+1}, associated with the sum or product of the two others, are the same as those of ℓ_n and ℓ_{n-1}. This has immediate consequences for the design of stabilized large graphs, since for given K the eigenvalues of $I \otimes A + \ell_{n+1} \otimes K$ depend on the eigenvalues of ℓ_{n+1} which are the same as those of ℓ_n and ℓ_{n-1}. In particular, if K is chosen to stabilize Equation 6.2 for both ℓ_n and ℓ_{n-1} then the same K will stabilize the product.

In fact, it is easy to see that, if g_n and g_{n-1} each have exactly one reach (and if the gluing is done by using a leader), one can calculate the eigenvectors of $g_n \cup_{j_n=j_{n-1}} g_{n-1}$ and $g_n \times_{j_{n-1}} g_{n-1}$ in terms of the eigenvectors of g_n and g_{n-1}. Since for the purposes of this paper we are mostly concerned with the spectrum of the Laplacian, we will not discuss these results here.

8. ORIENTABLE FLOCKS, A NONLINEAR MODEL

Note that the in formation solutions so far do not allow the formation itself to change direction. In other words, once agents start out flying in formation, say in a "V" shaped formation as geese, the tip of the "V" will always point in the same direction, even though the flock may move in the opposite direction. To make our model biologically a little more realistic we will briefly discuss how to overcome this problem by introducing a rotational term into the equations. To simplify the discussion, we will, in this section, assume that the communication graph has but a single reach and no independent leaders. The existence of certain stable rotating solutions, and other unusual solutions, is still an open problem. These are of interest to biology (see ref. 20) and we briefly address that question at the end this section. In contrast to the previous sections, Equation 8.1 is nonlinear. Thus the stability we prove in Theorem 8.7 is local.

We have already observed that positions are only calculated relative to the positions of other agents. Let us now assume that $d = 2$ and the desired relative positions are known up to an angle, namely the angle of flight.

The equations we will set up have a singularity whenever the speed of any of the agents is zero. So we will from now on assume that that is not the case. In the following E_i is the N by N matrix whose only non-zero entry is a one in the i-th diagonal entry. The vectors h_i are as in Section 2.

Definition 8.1. Let $z(t)$ be the position-velocity vector of a flock whose N members have velocities $\{v_i\}_{i=1}^N$. Let $R_v : \mathbb{R}^2 \to \mathbb{R}^2$ be the operation that rotates the x-axis (or e_1) onto the v axis. Now define the linear operator:

$$\mathcal{R}_z : X \to X \equiv h \to \sum_{i=1}^{N} E_i \otimes R_{v_i} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} h_i,$$

Definition 8.2. An orbit $\phi : \mathbb{R} \to X$ of the flock is said to be *in (oriented) formation* if it is given by

$$\phi_\alpha(t) = \mathcal{R}_{\mathbf{1}\otimes\alpha} h + \mathbf{1} \otimes \alpha \quad \text{and} \quad \dot{\alpha} = A\alpha \quad (\text{and} \quad A_4 = 0).$$

Denote this orbit by ϕ_α. This formation is oriented along the line of flight (as opposed to Definition 2.1).

Later on we will use the fact that our assumptions imply that $\mathcal{R}_{\mathbf{1}\otimes\alpha} h$ is a constant vector. this can be seen by noting that $\mathcal{R}_{\mathbf{1}\otimes\alpha} h$ only depends on the direction of the velocity part p of α. Since $A_4 = 0$, we know by Proposition 4.3 that p is constant.

Definition 8.3. With all the conventions the same as before (in addition to the above definition), the equations for an orientable flock are:

$$\dot{z} = I \otimes Az + L \otimes K(z - \mathcal{R}_z h). \tag{8.1}$$

The following result is obtained by substituting the in formation solution into Equation 8.1.

Lemma 8.4. For every position vector h and every solution α of $\dot{\alpha} = A\alpha$, the system of Definition 8.1 admits an (oriented) in formation solution ϕ_α.

The stability of the in formation solutions is a much more delicate issue here, because Equation 8.1 is non-linear. For simplicity, we call Equation 8.1 stable at a solution ϕ if upon substituting $z = \phi + y$ and linearizing, the resulting equation for y, $\dot{y} = My$, is such that the eigenvalues of M either have negative real part or are identical to zero. Take a particular in formation solution:

$$\phi_\alpha(t) = \mathcal{R}_{\mathbf{1}\otimes\alpha} h + \mathbf{1} \otimes \alpha \quad \text{where} \quad \alpha = q \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + p \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and suppose $A_4 = 0$ (so that p is constant). We will show that if $||p||$ is large enough, the system is stable at this solution. We first need two lemmas.

Lemma 8.5. Suppose $p \in \mathbb{R}^2$ fixed, and $v \in \mathbb{R}^2$ small. Then

$$\frac{p+v}{||p+v||} - \frac{p}{||p||} = \frac{1}{||p||}\left(v - \frac{(p,v)p}{||p||^2}\right) + \mathcal{O}(||v||^2) = \frac{1}{||p||}T_{p^\dagger}v + \mathcal{O}(||v||^2),$$

where $T_{p^\dagger} : \mathbb{R}^2 \to \mathbb{R}^2$ is the orthogonal projection onto the orthogonal complement of p.

Proof. This is of course well-known and can be verified by direct computation, or, more easily, geometrically: Project the segment in $\mathbb{R}^d$ connecting $p+x$ to p onto the hyperplane that is tangent to the unit sphere and orthogonal to p and finally divide by $||p||$. ∎

Lemma 8.6. Suppose $y = \sum_{i=1}^N e_i \otimes \left(x_i \otimes \begin{pmatrix}1\\0\end{pmatrix} + v_i \otimes \begin{pmatrix}0\\1\end{pmatrix}\right)$ and α as before. From Definition 8.1 we obtain:

$$\mathcal{R}_{\mathbf{1}\otimes\alpha+y}h - \mathcal{R}_{\mathbf{1}\otimes\alpha}h = \sum_{i=1}^N \frac{||h_i||}{||p||} e_i \otimes R_{h_i} T_{p^\dagger} v_i \otimes \begin{pmatrix}1\\0\end{pmatrix} + \mathcal{O}(||v||^2).$$

Here e_i denotes the standard i-th unit vector.

Proof. This follows by applying the previous Lemma to the terms of $\mathcal{R}_{\mathbf{1}\otimes\alpha+y}h - \mathcal{R}_{\mathbf{1}\otimes\alpha}h$ that correspond to the individual agents. Each term corresponds to the difference between rotating h_i by $\frac{p+v_i}{||p+v_i||}$ and rotating h_i by $\frac{p}{||p||}$. ∎

Using the methods of the previous sections we can choose K_3 and K_4 (see Remark 4.1) such that the system of Equation 2.3 is stabilized (see Definition 4.3). According to the proof of Theorem 4.4, this means that the matrix $I \otimes A + L \otimes K$ all of whose non-zero eigenvalues have negative real part. With some abuse of notation we will simply say that in this case the system of Equation 8.1 is stabilized.

Theorem 8.7. Suppose the flock has a communication graph with a single reach. Let ϕ_α be an in formation solution of Equation 8.1 with associated velocity p. Assume that the system is stabilized and that $A_4 = 0$. Then if $||p||$ is large enough (and all other parameters are held fixed), Equation 8.1 is stable at ϕ_α.

Proof. Substitute $\mathcal{R}_{\mathbf{1}\otimes\alpha}h + \mathbf{1}\otimes\alpha + y = \phi_\alpha + y$ for z in Equation 8.1. Note that the odd columns of A are zero so that $I \otimes A(\mathcal{R}h) = 0$. Also

$(L\otimes K)(\mathbf{1}\otimes\alpha)=0$. Finally, note that $\mathcal{R}_z h=\mathcal{R}_{\mathbf{1}\otimes\alpha+y}h$ because $\mathcal{R}_z$ depends only on the velocity-components of z. We then obtain from 8.1:

$$\mathbf{1}\otimes(\dot\alpha-A\alpha)+\dot y=(I\otimes A+L\otimes K)y+L\otimes K(\mathcal{R}_{\mathbf{1}\otimes\alpha}h-\mathcal{R}_{\mathbf{1}\otimes\alpha+y}h).$$

The left hand side reduces to $\dot y$ by the definition of in formation solutions above.

The first term on the right hand side is of the form My where M is a matrix whose eigenvalues have either negative real part or are identical to zero. Now linearize the second term of the right hand side so that it has the form Qy. Then from the foregoing, one concludes that the matrix Q has entries that are $\mathcal{O}(||p||^{-1})$, which for $||p||$ big are arbitrarily small. Thus the eigenvalues of M that had negative real part uniquely correspond to eigenvalues of $M+Q$ with negative real part (by continuity).

We still have to prove that zero eigenvalues of M remain zero after the perturbation. It suffices to prove that M has a zero eigenvalue with multiplicity $2d=4$. So fix ϕ_α and let $y(t)$ be the difference between two in formation solutions ϕ_β and ϕ_α (a 4-dimensional linear space). We calculate the derivative of the righthand side of Equation 8.1 at ϕ_α in the direction of $y(t)$. Substitute $\phi_\beta-\phi_\alpha$ for y in

$$\dot y=(I\otimes A+L\otimes K)y+L\otimes K(\mathcal{R}_{\mathbf{1}\otimes\alpha}h-\mathcal{R}_{\mathbf{1}\otimes\alpha+y}h).$$

Because $\mathcal{R}_z$ only depends on the velocity components of z, we have that

$$\mathcal{R}_{\mathbf{1}\otimes\alpha+y}=\mathcal{R}_{\mathbf{1}\otimes\alpha+(\mathbf{1}\otimes\beta-\mathbf{1}\otimes\alpha)}=\mathcal{R}_{\mathbf{1}\otimes\beta}.$$

Using that $L\mathbf{1}=0$ we find that the remaining Laplacian terms cancel and

$$\dot y=I\otimes Ay.$$

However, since $A_4=0$, A is nilpotent, and thus $(I\otimes A)^2=I\otimes A^2=0$. Therefore y corresponds to a vector in the eigenspace of L associated with the eigenvalue 0 of M. ∎

In fact, in the last part of this proof $y(t)\equiv\phi_\beta(t)-\phi_\alpha(t)$ is an affine function of time according to Lemma 8.4 (since $A_4=0$). Thus the Jordan block in question has dimension 2.

To obtain rotating solutions we follow the strategy of Theorem 5.2 and use the same form of the matrix A_4 as we did there. We first need a lemma.

Lemma 8.8. Suppose $||p||$ is constant, then

$$\frac{d}{dt}\mathcal{R}_{\mathbf{1}\otimes\alpha} = \frac{||\dot{p}||}{||p||}\mathcal{R}_{\mathbf{1}\otimes\dot{p}\otimes\begin{pmatrix}0\\1\end{pmatrix}}.$$

Proof. Set

$$y(t) = \sum e_i \otimes [q(t+dt) - q(t)] \otimes \begin{pmatrix}1\\0\end{pmatrix} + e_i \otimes [p(t+dt) - p(t)] \otimes \begin{pmatrix}0\\1\end{pmatrix},$$

and note that

$$\frac{d}{dt}\mathcal{R}_{\mathbf{1}\otimes\alpha} = \lim_{dt\to 0}\frac{1}{dt}\left(\mathcal{R}_{\mathbf{1}\otimes\alpha(t+dt)} - \mathcal{R}_{\mathbf{1}\otimes\alpha(t)}\right) = \lim_{dt\to 0}\frac{1}{dt}\left(\mathcal{R}_{\mathbf{1}\otimes\alpha(t)+y} - \mathcal{R}_{\mathbf{1}\otimes\alpha(t)}\right).$$

By hypothesis $||p||$ is constant and thus p is orthogonal to $\dot{p}$. In Lemma 8.6, $T_{p^\dagger}\dot{p} = \dot{p}$, and thus this lemma gives us that

$$\frac{d}{dt}\mathcal{R}_{\mathbf{1}\otimes\alpha}\,h = \sum_{i=1}^{N}\frac{||h_i||}{||p||}\,e_i \otimes \left(R_{h_i}\dot{p}\right) \otimes \begin{pmatrix}1\\0\end{pmatrix} = \sum_{i=1}^{N}\frac{||\dot{p}||}{||p||}\,e_i \otimes \left(R_{\dot{p}}h_i\right) \otimes \begin{pmatrix}1\\0\end{pmatrix},$$

which yields the desired result. ∎

Proposition 8.9. Under the coordinate change:

$$y \equiv z - \mathcal{R}_{\mathbf{1}\otimes\alpha}h - \mathbf{1}\otimes\alpha$$

Equation (8.1) becomes

$$\dot{y} = -|k|\mathcal{R}_{\mathbf{1}\otimes\dot{p}\otimes\begin{pmatrix}0\\1\end{pmatrix}}h + (I\otimes A + L\otimes K)\,y + L\otimes K\left(\mathcal{R}_{\mathbf{1}\otimes\alpha}h - \mathcal{R}_{\mathbf{1}\otimes\alpha+y}h\right). \tag{8.2}$$

Proof. This is the same substitution as in Theorem 8.7, but now instead of $p =$ constant and so $\frac{d}{dt}\mathcal{R}_{\mathbf{1}\otimes\alpha} = 0$, we apply the above Lemma. Furthermore, we recall that since $\dot{\alpha} = A_4\alpha$, the flock must describe a circular motion in which $q(t) = q_0(\cos kt, \sin kt)$, so that $||\dot{p}||/||p|| = |k|$. ∎

To check if we can get an in formation solution (with $k \neq 0$), we see if $y = 0$ is solution of Equation 8.2 in the previous Proposition. Clearly this is not the case. However, it is easy to convince oneself (heuristically) that if $||p||$ is large enough and $|k|$ small enough (and other parameters are held fixed) there are solutions that stay close to the in formation orbit (which is not itself a solution). It is be sufficient to argue that $||y||$ remains small for all time. When $|k| = 0$, the system drives itself exponentially fast to an in formation solution. When $|k|$ is small enough the perturbation cannot beat the exponential stability and y must remain close to zero. A more detailed analysis of this situation will be given in a forthcoming work. For now we illustrate these motions with a simulation. In the Fig. 1, we simulated a flock flying in a straight line, when the curvature k is turned on. The value for k is the circular part is 0.1, and f and g have the value of -8 and -12, respectively. The flock turns maintaining the configuration approximately constant, but aligned with the line of flight. In ref. 28, we look at these and other examples in more detail.

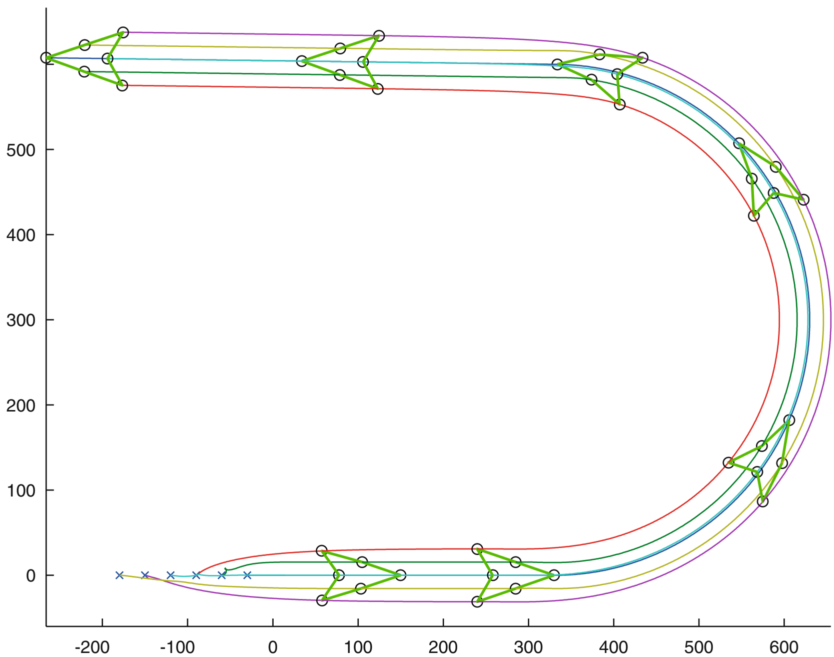

Fig. 1. (color online) This figure is a simulation using Equation 8.1 of a flock turning around. Note that the orientation of the flock's configuration changes. Each 'christmas tree shaped' hexagon is a snapshot of the position of the flock, the lines that form the figure only facilitate visual inspection, they have no physical or mathematical content.

In particular in biological applications, one is most interested in the situation where there are leaders. The theory for leaders in the current context can be developed parallel to the one in Section 6. However, the analysis of stable orbits is much more difficult in this case.

9. CONCLUDING REMARKS

We briefly summarize the main lines of thought in this paper. We study the dynamics of Equation 2.3 with the underlying communication graph being fixed but otherwise essentially completely arbitrary (looplessness being a natural requirement in this setting, not really a restriction). The effort is not directed at finding solutions—after all, the system of equations is *linear*—but rather at characterizing the stable set of solutions in the general case. So far, this had only been done in the case where the underlying graph has a single reach (that is: the underlying graph has a spanning tree).

The motivation comes on the one hand from the search for algorithms to control the flight of collections of artificial objects moving in formation. On the other hand aggregational behavior is also very common in biology. In the latter case, experiments and observations [(19)] suggest that at least in many cases, fish when swimming in a school change neighbors frequently. Furthermore these experiments also suggest that the formation is 'noisy'. Our analysis should be considered only as a preliminary step in the modeling of these phenomena. Such a step is justified by the fact—as this article demonstrates—that the analysis is substantial and indeed leads to unexpected new mathematical territory (the characterization of the multiplicity of the eigenspace of zero of the Laplacian of a graph in terms of the connectivity of the underlying graph, see ref. 3).

In reality, animals, when they get too close or too far away from another, will behave nonlinearly (they avoid the collision in one case, and simply cease to notice one another in the other). However, when they are in formation or very close to it, their flight may very well be modeled by linearized equations similar to the system we studied. One can of course add noise to the system here described. Our stability results are relevant in that the stabler a particular solution is, the larger the perturbation tends to need to be to knock the orbit away from the originally stable solution. This is of course the reason why the calculation of the eigenvalues closest to the origin is of paramount importance. Throughout the paper we use graph theoretic methods to aid in the calculation of these eigenvalues (see for example Section 7).

In Section 8 we study a system of equations (Equation 8.1) which is nonlinear for a different reason. It describes the flight of a flock (in $\mathbb{R}^2$)

with a tendency to orient the configuration along the line of flight. The linear model studied earlier serves as a good guideline for the behavior of this system. Of course, due to the nonlinearity, stability analysis gives only local results: a large enough perturbation may knock the system out of a stable in formation orbit.

Techniques for dealing with variable graphs exist,[10,21,22] mostly in the context of consensus seeking. The general tenor of those results is that if the varying graph has a spanning tree often enough, then consensus will be achieved. In principle this approach can be applied to our system, whether linear or nonlinear, and presumably with similar results. This will be done elsewhere.

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The authors fondly recall a recent and memorably festive visit of Mitchell to Portland State. There was no end to oysters, good wine, and interaction between faculty of about half a dozen departments.

One of the authors (JJPV) has many dear memories, first as a student at Cornell, and later as a postdoc with Mitchell at Rockefeller, of many inspiring evenings, always marked by Mitchell's unfailingly inquisitive and analytic nature and superb taste in wine.

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